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On p-Vectors of 3-Polytopes

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ABSTRACT

If P is a simple 3-dimensional polytope and p_i is the number of i sided faces of P then

$$p_6 \geqslant 2 + \frac{p_3}{2} - \frac{p_5}{2} - \sum_{j=7}^{n} p_i$$

where n is the largest number of edges on any face of P. Generalizations to non-simple polytopes and polytopes of higher genus are obtained.

A 3-polytope is a 3-dimensional set which is the convex hull of a finite number of points. By the p-vector of a 3-polytope P we mean a vector $(p_3, \ldots, p_n)$ where p_i is the number of i-sided faces of P and no face of P has more than n sides. The problem of characterizing the p-vectors seems quite hopeless and even, if we restrict ourselves to simple (i.e., 3-valent) 3-polytopes, the problem seems quite difficult.

A well-known result of Euler's formula is that, if $(p_3, \ldots, p_n)$ is the p-vector of a simple 3-polytope, then

$$\sum_{i=3}^{n} (6 - i) p_i = 12. \tag{1}$$

The following theorem is due to Eberhard [1]. *If $(p_3, p_4, p_5, p_7, \ldots, p_n)$ is a sequence satisfying* (1), *then there exists a value for p_6 such that $(p_3, \ldots, p_n)$ is the p-vector of some simple* 3-*polytope.*

In this paper we prove

THEOREM 1. *If $(p_3, \ldots, p_n)$ is the p-vector of a simple 3-polytope P then*

$$p_6 \geqslant 2 + \frac{p_3}{2} - \frac{p_5}{2} - \sum_{j=7}^{n} p_j$$

when

$$\sum_{j=7}^{n} p_j \geqslant 3. \tag{2}$$

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PROOF: We shall use the following well-known consequences of Euler's formula: If P is a simple 3-polytope with V vertices, E edges, and F faces then

$$V = 2F - 4, \tag{3}$$

$$\begin{aligned} &E \leqslant 3V - 6, &&\text{if} \quad V \geqslant 3, \\ &E \leqslant 1, &&\text{if} \quad V = 2, \\ &E = 0, &&\text{if} \quad V = 1 \quad \text{or} \quad 0. \end{aligned} \tag{4}$$

We obtain (3) from the equations $V - E + F = 2$ (Euler's formula) and $E = 3V/2$. We obtain (4) by observing that the maximum number of edges is attained when each face of the graph is triangular, in which case $E = 3F/2$. The result now follows from Euler's formula.

A face of P will be called *critical* provided it has at least seven edges and an edge of P will be called *critical* provided it is incident to two critical faces. For each critical face F we define $\delta(F)$ to be the number of critical faces meeting F. For each vertex v of P we define $\varphi(v)$ to be the number of critical edges meeting v. We now consider $\sum \delta(F)$ taken over all critical faces of P and $\sum \varphi(v)$ taken over all vertices of P. Each sum counts the critical edges twice and thus we have

$$\sum \delta(F) = \sum \varphi(v). \tag{5}$$

We now count the number of vertices of P. This is at least

$$\sum_{i \geqslant 7} ip_i - V_2 - 2V_3$$

where V_2 is the number of vertices which meet exactly two critical faces and V_3 is the number of vertices which meet three critical faces.

However, $V_2 + 2V_3 \leqslant \sum \varphi(v)$ and thus using (3) and (5) we have

$$2 \sum_{i \geqslant 3} p_i - 4 = V \geqslant \sum_{i \geqslant 7} ip_i - \sum \delta(F).$$

We now construct a new graph G as follows: the vertices of G correspond to the faces of P and two vertices are joined by an edge provided the corresponding faces of P meet. The sum $\sum \delta(F)$ is two times the number of edges in G and thus using (4) and, assuming that $\sum_{i \geqslant 7} p_i \geqslant 3$, we have

$$\sum \delta(F) \leqslant 6 \sum_{i \geqslant 7} p_i - 12,$$

$$2 \sum_{i \geqslant 3} p_i - 4 \geqslant \sum_{i \geqslant 7} ip_i - 6 \sum_{i \geqslant 7} p_i + 12, \tag{5}$$

$$2 \sum_{i \geqslant 3} p_i \geqslant 16 + \sum_{i \geqslant 7} (i - 6)\, p_i\, .$$

Using (4) we have

$$2 \sum_{i \geqslant 3} p_i \geqslant 4 + 3p_3 + 2p_4 + p_5;$$

thus

$$p_6 \geqslant 2 + \frac{p_3}{2} - \frac{p_5}{2} - \sum_{i \geqslant 7} p_i .$$

If we assume that $\sum_{i \geqslant 7} p_i = 2$ then we obtain

$$p_6 \geqslant -3 + \frac{p_3}{2} - \frac{p_5}{2};$$

but using (1) we see that

$$-3 + \frac{p_3}{2} - \frac{p_5}{2} < 0,$$

thus we have no bound for p_6. We also obtain no bound if $\sum_{i \geqslant 7} p_i < 2$.

Grünbaum [2] conjectured that, given a sequence $(p_3, p_4, p_5, p_7, ..., p_n)$ satisfying (1), there is a value of $p_6 < n$ such that $(p_3, ..., p_n)$ is the p-vector of some simple 3-polytope. Theorem 1 shows this to be false for, if $p_3 = 10t + 4$ and $p_{3t} + 6 = 10$, then $p_6 \geqslant 5t - 6$.

It can even be shown that Grünbaum's conjecture is false if n is replaced by any function of n. Suppose we can choose $p_6 \leqslant f(n)$. Let $(p_3, ..., p_n)$ be a p-vector with

$$p_3 = 4 + \frac{n-6}{3} p_n$$

and $p_i = 0$ for all $i \neq 3, n$. By (2), we have

$$p_6 \geqslant \frac{p_3}{2} - p_n + 2,$$

$$p_6 \geqslant 2 + \frac{n-6}{6} - p_n + 2,$$

$$p_6 \geqslant 4 + \frac{n}{6} p_n;$$

thus

$$p_n \leqslant \frac{6(p_6 - 4)}{n} \leqslant \frac{6(f(n) - 4)}{n} .$$

So by choosing p_n larger than $6(f(n) - 4)/n$ we reach a contradiction.

From (5) we may derive the following alternate form for inequality (1):

$$p_6 \geqslant 8 - \sum_{i=3}^{5} p_i + \frac{1}{2} \sum_{i=7}^{n} (i - 8) p_i . \tag{6}$$

In this form the inequality is also valid for 3-polytopes which are not necessarily simple.

If P is any 3-polytope we replace each vertex of valence greater than three by a circuit and then remove an arbitrary edge from each of these circuits (see Figure 1). This gives us a graph of a simple 3-polytope P' and

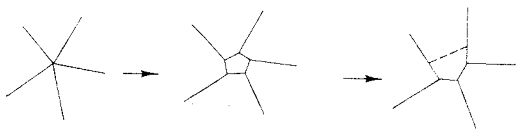

FIGURE 1

in producing P' no face has decreased in number of edges. If we let the symbols V' and p_i' refer to P' and V and p_i refer to P then by the reasoning in Theorem 1 we have

$$2 \sum_{i \geqslant 3} p_i = 2 \sum_{i \geqslant 3} p_i' \geqslant 16 + \sum_{i \geqslant 7} (i - 6) p_i' \geqslant 16 + \sum_{i \geqslant 7} (i - 6) p_i ,$$

from which we conclude (6).

This inequality is valid provided

$$\sum_{i \geqslant 7} v_i + \sum_{i \geqslant 7} p_i \geqslant 3,$$

where v_i is the number of i-valent vertices of P.

REMARKS

1. If we consider 3 valent 3-connected graphs in a manifold of genus g the result is

$$2 \sum_{i<7} p_i \geqslant \sum_{i \geqslant 7} (i - 8) p_i + 16(1 - g).$$

2. The author is indebted to Branko Grünbaum who suggested many refinements and simplifications for the author's original results.

3. A simpler proof of Eberhard's theorem may be found in [2]; also there are interesting results concerning special 3-polytopes in [2].

4. Grünbaum has recently proved that, if $p_3 = p_4 = 0$ and (1) is satisfied, then there is a value of $p_6 \leqslant 8$ for which $(p_3, \dots, p_n)$ is the p-vector of a simple 3-polytope.

References

1. V. Eberhard, *Zur Morphologie der Polyeder*, Teubner, Leipzig, 1891.
2. B. Grünbaum, *Convex Polytopes*, Wiley, New York, 1967.